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Local Inversion Theorems without Assuming Continuous Differentiability

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In [6] it is proved that if $D \subset R^n$ is open and $f: D \to R^n$ is Fréchet differentiable on D and $f'(x) \in \operatorname{Isom}(R^n, R^n)$ for every $x \in D$, then f is a local diffeomorphism on D. We shall extend this theorem for infinite dimensional Banach spaces E and mappings $f: D \to E$, $D \subset E$ open, $f = I - T$, where $T: D \to E$ is compact and $I: D \to E$, $I(x) = x$ for every $x \in D$. We shall also prove a generalization of the Hadamard–Levi theorem for such mappings. Our methods are purely analytical, since the single instrument we are using for establishing the results of the present paper is the topological degree theory. Our methods are different from [6] even in the finite dimensional case, since we not use the results from [8].

DEFINITION 1. Let $f: E \to F$. We denote by $B_f = \{x \in E \mid f$ is not a local homeomorphism at $x\}$.

DEFINITION 2. Let E be a Banach space and $M \subset E$. We denote by $K_1(M)$ the set of all mappings $T: M \to E$, $T = I - K$, where $K: M \to E$ is compact. If M is open and $\dim E < \infty$, then $K_1(M) = C(M, E)$. If D is open, bounded and $T \in K_1(\bar{D})$, then T is closed and proper [5, p. 80; 7, p. 69].

LEMMA 1. *Let E be a complete metric space, $K \subset E$ closed, $f: E \to F$ continuous, open, and discrete, and $M = \{x \in K \mid$ there exists $U \subset E$ open such that $x \in U$ and for every $z \in U \cap K$, $\{z\} = f^{-1}(f(z)) \cap U\}$. Then M is densely in K.*

Proof. For every $x \in E$ we define $r(x) = \sup\{r \in R \mid B(x, r) \cap f^{-1}(f(x)) = \{x\}\}$. We denote by $\{r_n\}_{n \in N}$ the set of all rational numbers and let $K_n = \{x \in E \mid r(x) \geqslant r_n\}$. We prove that K_n is closed for every $n \in N$. Indeed, let $x \in E$ and $x_k \in K_n$ be such that $x_k \to x$. Suppose that $x \notin K_n$ and

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let $\varepsilon > 0$ be such that $r(x) < r_n - \varepsilon$. Then we can find $z \in E$ such that $d(x, z) < r_n - \varepsilon$ and $f(x) = f(z)$. Since f is open, we can find $U_1 \in V(x)$, $U_2 \in V(z)$ such that $U_1 \subset B(x, \varepsilon/2)$, $U_2 \subset B(z, \varepsilon/2)$, and $f(U_1) = f(U_2) = V \in V(f(x))$. There exists $p \in N$ such that $x_p \in U_1$, hence there exists $z_p \in U_2$ such that $f(x_p) = f(z_p)$. Then $d(x_p, z_p) < r_n$, which represents a contradiction, since $x_p \in K_n$. We proved that K_n is closed for every $n \in N$. Let now $U \subset E$ be open such that $K \cap U \neq \varnothing$. Then $\overline{K \cap U} = (\overline{K \cap U} \setminus K \cap U) \cup \bigcup_{n=1}^{\infty} K_n \cap K \cap U$. Since K is closed, it results that $\overline{K \cap U} \cap U = K \cap U$, hence $\overline{K \cap U} \setminus K \cap U$ is closed in $\overline{K \cap U}$ and the interior of $\overline{K \cap U} \setminus K \cap U$ in $\overline{K \cap U}$ is empty. Since $\overline{K \cap U}$ with the induced topology is of the second Baire category, we can find $n \in N$ such that the interior of $\overline{K_n \cap K \cap U}$ in $\overline{K \cap U}$ is not empty. Let $Q_1 \subset E$ open be such that $\overline{K \cap U} \cap Q_1 \neq \varnothing$ and $\overline{K \cap U} \cap Q_1 \subset \overline{K_n \cap K \cap U}$. Let $x \in \overline{K \cap U} \cap Q_1$. Then there exists $x_k \in K \cap U$ such that $x_k \to x$, hence $x_k \in Q_1$ for k great enough, hence we can find $Q \subset U \cap Q_1$ a ball of radius smaller than $r_n/2$ and such that $K \cap Q \neq \varnothing$. Then $K \cap Q \subset K \cap U \cap Q_1 \subset \overline{K \cap U} \cap Q_1 \subset \overline{K_n \cap K \cap U}$ and since K_n is closed, it results that $K \cap Q = K_n \cap K \cap Q$. It is obvious that for every $z \in K \cap Q$, $f^{-1}(f(z)) \cap Q = \{z\}$ and the theorem is proved.

THEOREM 1. *Let E be a Banach space, $D \subset E$ open, and $f \in K_1(D)$. Suppose that f is differentiable and $f'(x) \in \operatorname{Isom}(E, E)$ for every $x \in D$. Then f is a local homeomorphism on D.*

Proof. We can suppose that $D = E$ and we show that f is open at every $p \in E$. Let $p \in E$ be fixed and $m_p = \inf_{\|q\|=1} \|f'(p)(q)\|$. Then $m_p > 0$. We can find $r_p > 0$ such that for $\|q - p\| \leqslant r_p$, $\|f(q) - f(p) - f'(p)(q - p)\| < m_p \|q - p\|$. Then, for $\|q - p\| \leqslant r_p$, we have $\|f(q) - f(p) - f'(p)(q - p)\| < m_p \cdot \|q - p\| \leqslant \|f'(p)(q - p)\|$. We obtained that

$$\|f(q) - f(p) - f'(p)(q - p)\| < \|f'(p)(q - p)\| \qquad \text{for} \quad \|q - p\| \leqslant r_p. \tag{1}$$

We remark that $f'(p) \in K_1(E)$ (see [7, p. 68]) and that for every $q \in E$, $i(f'(p), q, f'(p)(q)) = i_p = \pm 1$ (see [7, p. 75] or [5, p. 66]). We consider the compact homotopy H: $\bar{B}(p, r_p) \times [0, 1] \to E$, $H(q, t) = f'(p)(q - p) + t(f(q) - f(p) - f'(p)(q - p))$. Then, for $\|q - p\| = r_p$, $H(q, t) \neq 0$ for every $t \in [0, 1]$, hence (see [7, p. 72] or [5, p. 61]), $i(f, p, f(p)) = d(f, B(p, r_p), f(p)) = d(f - f(p), B(p, r_p), 0) = d(f'(p) - f'(p)(p), B(p, r_p), 0) = d(f'(p), B(p, r_p), f'(p)(p)) = i(f'(p), p, f'(p)(p)) = i_p = \pm 1$. We proved that $i(f, p, f(p)) = i_p = \pm 1$. We prove now that f is open at p. Indeed, let $r_1 < r_p$. Then $f \mid \bar{B}(p, r_1)$ is closed and from (1), $f^{-1}(f(p)) \cap \bar{B}(p, r_1) = \{p\}$. Let $V \in V(f(p))$ be a ball such that $V \cap f(S(p, r_1)) = \varnothing$. Then, for every $q \in V$, $d(f, B(p, r_1), q) = d(f, B(p, r_1), f(p)) = i_p \neq 0$, hence $f(B(p, r_1)) \supset V$ (see [5, p. 60] or [7, p. 71]) and f is open at p. We proved that f is open and discrete at every $p \in E$. From Lemma 1, $\operatorname{int} B_f = \varnothing$.

Suppose that $B_f \neq \emptyset$. Let $U \subset E$ be open and bounded such that $U \cap B_f \neq \emptyset$. If we apply Lemma 1 for $K = B_f$, we can find $Q_1 \subset U$ open such that $B_f \cap Q_1 \neq \emptyset$ and for every $u \in B_f \cap Q_1$, $f^{-1}(f(u)) \cap Q_1 = \{u\}$. Let $v \in B_f \cap Q_1$ and $Q_2 \in V(v)$ be such that $\bar{Q}_2 \subset Q_1$. Then $f(v) \notin f(FrQ_2)$. Let $V_1 \in V(f(v))$ be a ball such that $V_1 \cap f(FrQ_2) = \emptyset$ and $Q = f^{-1}(V_1) \cap Q_2$. Then $B_f \cap Q \neq \emptyset$ and if $x_1, x_2 \in B_f \cap Q$, $i(f, x_1, f(x_1)) = d(f, Q_2, f(x_1)) = d(f, Q_2, f(x_2)) = i(f, x_2, f(x_2))$. We finally found an open set $Q \subset E$ such that $B_f \cap Q \neq \emptyset$ and such that for every $v \in B_f \cap Q$, $f^{-1}(f(v)) \cap Q = \{v\}$ and $i(f, v, f(v))$ is constant. Suppose that $i(f, v, f(v)) = 1$ for every $v \in B_f \cap Q$. Let $A_{mn} = \{x \in B_f \cap Q \mid m_x > 5/n$ and $\|f(z) - f(x) - f'(x)(z-x)\| < (1/n)\|z-x\|$ for every $z \in E$ such that $\|z-x\| \leqslant 1/m\}$. Since $\overline{B_f \cap Q}$ with the induced topology is of the second Baire category and $\overline{B_f \cap Q} = (\overline{B_f \cap Q} \setminus (B_f \cap Q)) \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A_{mn}$, it results that there exist $m_0, n_0 \in N$ such that $\overline{A_{m_0 n_0}}$ has nonempty interior in $\overline{B_f \cap Q}$. Let $U_1 \subset E$ open be such that $U_1 \cap \overline{B_f \cap Q} \neq \emptyset$ and $U_1 \cap \overline{B_f \cap Q} \subset \overline{A_{m_0 n_0}}$. Then $U_1 \cap B_f \cap Q \neq \emptyset$ and we now fix $x \in B_f$ and $r > 0$ such that $B(x, r) \subset Q \cap U_1$, hence $B(x, r) \cap B_f \subset B_f \cap Q \cap U_1 \subset U_1 \cap \overline{B_f \cap Q} \subset \overline{A_{m_0 n_0}}$. Let $b \in B(x, r/4) \setminus B_f$ be fixed. Let $A = \{t \in R \mid S(b, t) \cap B_f \neq \emptyset\}$. Since $\|b - x\| \in A$, it results that $A \neq \emptyset$ and let $l = \inf A$. Then $0 < l \leqslant r/4$ and let $m \in N$ be chosen such that $m \geqslant m_0$ and $1 < 2 \cdot m \cdot l$. Since $A_{m_0 n_0} \subset A_{m n_0}$, it results that $B(x, r) \cap B_f \subset \overline{A_{m n_0}}$. Let $s = \min(r/4, 1/8m)$. From the definition of l, we can find $a_1 \in B_f$ such that $l \leqslant \|b - a_1\| \leqslant l + s$. Since $\|x - a_1\| \leqslant \|b - x\| + \|b - a_1\| \leqslant r/4 + l + r/4 < r$, it results that $a_1 \in B(x, r) \cap B_f$. Since $A_{m n_0}$ is densely in $B(x, r) \cap B_f$, we can find $a \in A_{m n_0}$ such that $\|a - a_1\| < 1/(8 \cdot m)$, hence $l \leqslant \|b - a\| \leqslant \|b - a_1\| + \|a_1 - a\| \leqslant l + 1/8m + 1/8m = l + 1/4m$. We found $a \in A_{m n_0}$ such that $l \leqslant \|b - a\| < l + 1/4m$ and let us fix $z = a + (b-a)/(2 \cdot m \cdot \|b - a\|)$. Then $\|z - a\| = 1/2m$ and let $c = (2 \cdot \|z - a\|)/(m_a \cdot n_0 - 1)$ and $y \in E$ such that $\|y - z\| = c$. Since $m_a \cdot n_0 > 5$, it results that $2/(m_a \cdot n_0 - 1) < \frac{1}{2}$. Then $\|y - a\| \leqslant \|z - a\| + \|y - z\| \leqslant 3/4m$, hence $\|f(y) - f(z) - f'(a)(y - z)\| \leqslant \|(f(y) - f(a) - f'(a)(y - a)) - (f(z) - f(a) - f'(a)(z - a))\| \leqslant \|f(y) - f(a) - f'(a)(y - a)\| + \|f(z) - f(a) - f'(a)(z - a)\| < (1/n_0) \cdot (\|y - a\| + \|z - a\|) \leqslant (1/n_0)(2 \cdot \|z - a + c) = c \cdot m_a = m_a \cdot \|y - z\| \leqslant \|f'(a)(y - z)\|$. We obtained that $\|f(y) - f(z) - f'(a)(y - z)\| < \|f'(a)(y - z)\|$ for every $y \in E$ such that $\|y - z\| = c$. By the same argument as in (1), we find that $d(f, B(z, c), f(z)) = d(f'(a), B(z, c), f'(a)(z)) = i_a = i(f, a, f(a)) = +1$ since $a \in B_f \cap Q$. Also, $\|b - a\| \geqslant l > 1/2m$, hence $1 - 1/(2m \cdot \|b - a\|) > 0$. Then

$$\|b - z\| = \left\| b - a - \frac{b - a}{2m \cdot \|b - a\|} \right\| = \left\| (b - a)\left(1 - \frac{1}{2m \cdot \|b - a\|}\right) \right\|$$

$$= \|b - a\| \cdot \left(1 - \frac{1}{2m \cdot \|b - a\|}\right) = \|b - a\| - \frac{1}{2m}.$$

Also, for $\|y-z\| \leqslant c$, we have $\|y-z\| \leqslant \frac{1}{2} \cdot \|z-a\| = 1/4m$, hence $\|b-y\| \leqslant \|b-z\| + \|z-y\| \leqslant \|b-a\| - 1/2m + 1/4m = \|b-a\| - 1/4m < l$, hence $B(z, c) \subset B(b, l)$ and f is a local homeomorphism on $B(b, l)$. Then $i(f, d, f(d))$ is constant for every $d \in B(z, c)$ and let $\bar{m} = i(f, z, f(z))$ and $\bar{n} = \operatorname{Card} f^{-1}(f(z)) \cap B(z, c)$. Then $f(z) \notin f(S(z, c))$ and $1 = d(f, B(z, c), f(z)) = \sum_{t \in f^{-1}(f(z)) \cap B(z,c)} i(f, t, f(t)) = \bar{m} \cdot \bar{n}$. Since $\bar{m} \cdot \bar{n} = 1$, we find that $\bar{m} = \bar{n} = 1$ and $i(f, z, f(z)) = 1$. Since $B(z, c) \subset B(b, l)$, it results that $i(f, b, f(b)) = 1$. Since b was arbitrary chosen in $B(x, r/4) \setminus B_f$, it results that $i(f, b, f(b)) = 1$ for every $b \in B(x, r/4)$. Let $0 < \varepsilon < r/4$ be such that $\bar{B}(x, \varepsilon) \cap f^{-1}(f(x)) = \{x\}$. Then we can find $V_2 \in V(f(x))$ a ball such that $V_2 \cap f(S(x, \varepsilon)) = \emptyset$. Let $M = B(x, \varepsilon) \cap f^{-1}(V_2)$ and $w \in M$. We denote $\bar{q} = \operatorname{Card} f^{-1}(f(w)) \cap B(x, \varepsilon)$. Then $1 = i(f, x, f(x)) = d(f, B(x, \varepsilon), f(x)) = d(f, B(x, \varepsilon), f(w)) = \sum_{t \in f^{-1}(f(w)) \cap B(x,\varepsilon)} i(f, t, f(t)) = \bar{q}$, hence $\bar{q} = 1$ and $f \mid M$ is injective. Since f is open, it results that f is a local homeomorphism at x, which represents a contradiction. The proof is the same if we suppose that $i(f, v, f(v)) = -1$ for every $v \in B_f \cap Q$. We finally proved that f is a local homeomorphism on E.

We shall generalize now a well-known global inversion theorem due to Hadamard and Levy. This result is in connection with [1, 4].

THEOREM 2. *Let E be a Banach space and $f \in K_1(E)$. Suppose that:*

(a) *f is differentiable and $f'(x) \in \operatorname{Isom}(E, E)$ for every $x \in E$;*

(b) *$\|[f'(x)]^{-1}\| \leqslant h(\|x\|)$ for every $x \in E$, $h: R_+ \to R_+ \setminus \{0\}$ continuous such that $\int_0^\infty ds/h(s) = \infty$.*

Then $f: E \to E$ is a global homeomorphism.

Proof. From Theorem 1, f is a local homeomorphism. Let $x_0 \in E$, $y \in E$, and $p: [0, 1] \to E$, $p(t) = (1-t) f(x_0) + ty$. Let $0 < a < 1$, $q: [0, a) \to E$ continuous such that $q(0) = x_0$, $f(q(t)) = p(t)$ for every $t \in [0, a)$. We consider $0 < b < c < a$, $\varepsilon > 0$, and $b = x_0 < x_1 < \cdots < x_n = c$, a partition Δ of $[b, c]$. Then, since $q \mid [b, c]$ is differentiable, for every $i \in \{0, 1, \ldots, n-1\}$ there exists $y_i \in [x_i, x_{i+1}]$ such that $\|q(x_{i+1}) - q(x_i)\| < (\|q'(y_i)\| + \varepsilon) \cdot |x_{i+1} - x_i|$. We obtain that

$$\begin{aligned} \|q(b) - q(c)\| &\leqslant \sum_{i=0}^{n-1} \|q(x_{i+1}) - q(x_i)\| \\ &\leqslant \sum_{i=0}^{n-1} (\|q'(y_i)\| + \varepsilon) \cdot |x_{i+1} - x_i| \end{aligned}$$

$$\leqslant \varepsilon \cdot \sum_{i=0}^{n-1} |x_{i+1} - x_i| + \|y - f(x_0)\|$$
$$\cdot \sum_{i=0}^{n-1} \|[f'(q(y_i)]^{-1}\| \cdot |x_{i+1} - x_i|$$
$$\leqslant \varepsilon \cdot |b - c| + \|y - f(x_0)\| \cdot \sum_{i=0}^{n-1} h(\|q(y_i)\| \cdot |x_{i+1} - x_i|.$$

Since we can choose $\|\Delta\|$ and ε arbitrarily small, we obtain that $\|q(b) - q(c)\| \leqslant \|y - f(x_0)\| \cdot \int_b^c b(\|q(t)\|)\, dt$. As in [1], we prove that there exists a unique path q: $[0, 1] \to E$ such that $q(0) = x_0$, $f(q(t)) = p(t)$ for $t \in [0, 1]$ and that f: $E \to E$ is a global homeomorphism.

Remark. Theorems 1 and 2 remain true if we ask f to be differentiable only on $E \backslash K$, where $K \subset E$ is countable. If $\dim E = \infty$ and f is light, Theorem 1 and Theorem 2 remain true if we ask f to be differentiable only on $E \backslash K$, where $K = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset E$ compact. It is interesting that K may be dense in E! For the proof we can use Theorem 6 from [3] and Theorem 1 from [2] and with a slight modification, the proof remains the same.

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